The work described in this thesis was carried out by me under the supervision of Dr. Menaka Liyanage and a report on this has not been submitted to any University for another degree.

Date: 26th October 2001

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(K.K.W.A. Sarath Kumara)

I/~~We~~ certify that the above statement made by the candidate is true and that this thesis is suitable for submission to the University for the purpose of evaluation.

Date: October 26th, 2001

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LINKS BETWEEN PRIME IDEALS IN A SERIAL NOETHERIAN RING

by

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ABSTRACT

LINKS BETWEEN PRIME IDEALS IN A SERIAL NOETHERIAN RING

by

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Links between prime ideals defined originally for Noetherian rings were later defined for serial rings. It is quite natural to ask whether there is a link according to the definition† of links between prime ideals in a Noetherian ring from prime ideals Q to P whenever Q and P are linked according to the definition†† of links between prime ideals in a serial ring and the ring is given to be Noetherian. Here we prove that the answer is affirmative.

† If P and Q are prime ideals in a Noetherian ring R, then we say there is a *link* from Q *to* P, if there is an ideal A of R such that $QP \leq A < Q \cap P$ and $Q \cap P/A$ is non - zero and torsionfree as a right R/P - module and as a left R/Q - module.

†† If P and Q are prime ideals in a serial ring R, then we say that there is a *link* from Q *to* P, if (i) P and Q are equal or incomparable, and (ii) $QP \subset Q \cap P$.

CHAPTER 1

INTRODUCTION

The rings (modules) with ascending chain condition on right ideals (submodules) are called "Noetherian rings (modules)". The importance of these chain conditions was first demonstrated by Noether. Basic equivalence conditions of Noetherian modules are given in Chapter 2. In 1999 Tariq Rizvi and D.V. Huynh [TH] obtained a special characterization of Noetherian rings (without assuming the commutativity). It was proved there that a ring R is right Noetherian if and only if every cyclic right R-module is a direct sum of a projective module P and a module Q, where Q is either injective or Noetherian.

In 1927, Artin generalized rings, which satisfy both ascending and descending chain conditions for right ideals. In 1939, Hopkins obtained the consequence that every right Artinian ring is right Noetherian when the ring contains the multiplicative identity. It is well known that the converse is not true.

Semiprime ideals were introduced in the commutative case by Krull in 1929 and in the non-commutative case by Nagata in 1951. Krull also proved that a commutative ring is semiprime if and only if it has no non-zero nilpotent elements. The definition of prime ideals that use the product of ideals (i.e. $AB \leq P$ implies $A \leq P$ or $B \leq P$) was also introduced by Krull in both the commutative and the non-commutative cases. The existence of minimal primes (i.e. every prime ideal contains a minimal prime) was initially proved for the commutative case by Krull in 1929, and it is now known that this is also true for the non-commutative case as well. In 1951, Nagata proved finiteness of the set of minimal primes for a ring with the ascending chain condition on semiprime ideals.

Uniform right ideals were introduced by Goldie in 1960. Also, he proved in 1960 that a module has finite rank if and only if it contains no infinite direct sum of nonzero submodules.

The existence of regular elements in an essential right ideal was proved by Goldie initially for prime right and left Goldie rings in 1958, and for semiprime right Goldie rings in 1960. The concept of essential submodules was introduced by Johnson in 1951. However, the name was given by Eckmann and Schopf in 1953. In 1958, Goldie proved that a ring is right and left order in a simple Artinian ring if and only if R is a prime right and left Goldie ring.

It is well known that any commutative integral domain has a field of fractions. However, this need not to be true for a non-commutative ring lacking zero divisors. Ore in 1931 gave a necessary and sufficient condition for a division ring of fraction to exist. While many interesting ring theoretic results were proven in between, it is probably fair to say that the modern study of non-commutative Noetherian rings began with Goldie's work in 1958-1960 giving necessary and sufficient condition for a ring to have a semisimple ring of fractions. Goldie showed that a Noetherian ring has a semisimple ring of fractions if and only if it is semiprime and it has a division ring of fractions if and only if it has no zero divisors. This result is much deeper than Ore's 1931 result. Some results on semiprime Goldie rings are given in Chapter 3. In 1949, Asano proved that a ring has a right ring of fractions with respect to its regular elements if and only if its regular elements satisfy the right ore condition and later that a ring has a right ring of fractions with respect to a multiplicative set X of regular elements if and only if X satisfies the right ore condition. The general criterion using the Ore condition together with the reversibility condition was given by Gabriel in 1962.

The concept of a serial ring was initiated by Nakayama in 1941. In 1973, Warfield [W] defined serial modules. Also he proved in 1979, that a left and right Noetherian serial ring is a direct sum of Artinan rings and prime rings. Furthermore, Warfield showed that any uniform module over a serial Noetherian ring is uniserial (Theorem 2.13).

Bruno J. M ü ller contributed a lot into the theory of serial rings. He proved that a ring is semiperfect if and only if the multiplicative identity can be written as a sum of orthogonal, local idempotents implies that every serial ring is semiperfect. With Surjeeth Sing he investigated the relationships between prime ideals over serial rings, and introduced links [MS2]. In 1984, Sujeeth Singh proved that $P^2 = P$ for any non-maximal prime ideal P in a serial Noetherian ring [SS]. Using this we showed that in a serial Noetherian ring there are no self links for non- maximal primes.

In 1999, M ü ller and Guerriero investigated the prime ideals of a serial ring R in detail. They obtained an explicit description for the prime ideals of R, and an analysis of the forks of the spectrum in terms of the completely prime ideals. They also defined an associated semiprime ideal for any indecomposable injective R-module and proved that the associated semiprime ideal is either Goldie prime, or the Goldie semiprime intersection of a full fork [GM].

Ideal links (not named, but with the notation $\rightsquigarrow$) appeared in the work of Jateganoker in 1973. Second layer links (called links, and with the same notation $\rightsquigarrow$)

between prime ideals in a fully bounded Noetherian ring appeared in the work of M ü ller in 1976. In 1990 links between prime ideals in a serial ring appeared in a paper of M ü ller and Surjeet Singh [MS2].

Before getting into our main contribution, let us quote the following two definitions.

Definition: If P and Q are prime ideals in a serial ring R, then we say that there is a *link* from Q *to* P, write $Q \rightsquigarrow P$ if

(i) P and Q are equal or incomparable,

(ii) $QP \subset Q \cap P$.

Definition: If P and Q are prime ideals in a Noetherian ring R, then we say there is a *link* from Q *to* P, write $Q \rightsquigarrow P$, if there is an ideal A of R such that $QP \leq A < Q \cap P$ and $Q \cap P/A$ is non - zero and torsionfree as a right R/P - module and as a left R/Q - module. In this case $Q \cap P/A$ is called a *linking bimodule* between Q and P.

Main Theorem: Suppose P and Q are prime ideals in a serial Noetherian ring R. Then $Q \rightsquigarrow$ P if and only if $Q \rightsquigarrow P$.

Thus, both definitions are the same when the ring is given to be serial Noetherian.

In Chapter 4, we collect some results on prime ideals in serial Noetherian rings. At the end we give some results that we have proved. In Chapter 5 we describe the results obtained at the preliminary level. There we proved our main theorem with added hypothesis. These were done at the preliminary stage to clarify the general case.

CHAPTER 2

PRELIMINARIES

In this chapter we give the basic definitions and known results that are used in this report. Some examples are also added. These definitions, results and examples are based on [BJN], [CH], [LM], [M1], [SSZ] and [W].

We use the following proposition in defining a Noetherian module. Noether (1921) was the first person to demonstrate the significance of this result. We omit the proof of this proposition as it is well known.

Proposition 2.1 ([GW]-Proposition 1.1) For a module A, the following conditions are equivalent:

(a) A has ascending chain condition on submodules.

(b) Every non-empty family of submodules of A has a maximal element.

(c) Every submodule of A is finitely generated.

Definition: A module A is *Noetherian* if the equivalent conditions of Proposition 2.1 are satisfied.

A ring R is *right (left) Noetherian* if the right module R_R (left module ${}_R R$) is Noetherian. If both conditions hold, R is called a *Noetherian ring.*

Rephrasing Proposition 2.1 for the ring itself, we can see that a ring R is right (left) Notherian if and only if R has ascending chain condition on right (left) ideals, if and only if all right (left) ideals of R are finitely generated.

Definition: A module A is *Artinian* provided A satisfies the *descending chain condition* on submodules. i.e. there does not exist a properly descending infinite chain $A_1 > A_2 > ...$ of submodules of A.

A ring R is *right (left) Artinian* if and only if the right module R_R (left module ${}_R R$) is Artinian. If both conditions hold, R is called an *Artinian ring.*

Proposition 2.2: A is Artinian if and only if every non-empty family of submodules of A has a minimal element.

Proposition 2.3 ([GW]-Proposition 1.3): Let B be a submodule of a module A. Then A is Notherian if and only if B and A/B are both Noetherian.

Examples for Noetherian and Artinian Modules/ Rings:

1) Any finite-dimensional vector space V over a field K is Noetherian and Artinian as a K- module, because the properly of ascending chain of submodules (subspaces) of V cannot contain more than $\dim_K(V)+1$ subspaces.

2) The ring R of 2x2 matrices over Q of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ with $a \in Z$ and $b, c \in Q$ makes a ring, which is right Noetherian but not left Noetherian. (Note that for any non - negative integer k, $A_k = \left\{ \begin{pmatrix} 0 & m/2^k \\ 0 & 0 \end{pmatrix} \mid m \in Z \right\}$ is a left ideal of R. Further $A_k < A_{k+1}$ as $m/2^k = 2m/2^{k+1}$ and $\begin{pmatrix} 0 & 1/2^{k+1} \\ 0 & 0 \end{pmatrix} \notin A_k$. Thus we have a non-terminating strictly ascending chain $A_0 < A_1 < A_2 ...$ of left ideals in R.). This shows that R is not left Noetherian. But R is right Noetherian. (For, let A ($\neq 0$) be a right ideal in R and let $\alpha e_{11} + \beta e_{12} + \gamma e_{22} \in A$; where e_{ij} denotes the matrix with 1 in $(i, j)^{th}$ position and zero elsewhere, and $\alpha \in Z$, $\beta, \gamma \in Q$.

Claim(i): If $\alpha \neq 0$, then A is generated by matrices $\delta e_{11}, e_{12}, e_{22}$ or by δe_{11} and e_{12} where δ is the least positive integer such that $\delta e_{11} + b e_{12} + c e_{22} \in A$ for some $b, c \in Q$.

Proof: Since $(\delta e_{11} + be_{12} + ce_{22}) e_{12} \in A$ we get $\delta e_{12} \in A$. Therefore, $\delta e_{12}(1/\delta) e_{22} = e_{12} \in A$. Hence $be_{12} = e_{12}(be_{22}) \in A$. Also $(\delta e_{11} + be_{12} + ce_{22})e_{11} \in A$ gives $\delta e_{11} \in A$. Thus we get $ce_{22} \in A$. Now in case $c = 0$ for all such δ and b, A is generated by δe_{11} and e_{12}. Otherwise, $c e_{22}\left(\frac{1}{c}\right) e_{22} \in A$ and A is then generated by the matrices δe_{11}, e_{12} and e_{22}.

Claim (ii): If $\alpha = 0$, then A is generated by matrices e_{12} and e_{22} or by e_{12} alone.

Proof: If all the elements of A are of the type $\lambda(be_{12} + c e_{22})$ for some $\lambda \in Q$, then A is generated by matrices $be_{12} + c e_{22} \in A$. Otherwise, there exist $b_1, c_1 \in Q$ such that $b_1 e_{12} + c_1 e_{22} \in A$ but $b_1 c \neq b c_1$. If $(bc_1 - b_1 c)e_{12} \in A$ we get $(bc_1 - b_1 c)e_{12}\left(\frac{1}{(bc_1 - b_1 c)}\right)e_{22} \in A$ and hence, $e_{12} \in A$. Therefore, $e_{12} b e_{22} \in A$. Since $be_{12} = e_{12} b e_{22} \in A$, we have $ce_{22} \in A$. This implies either $c = 0$ or $e_{22} \in A$. Thus A is either generated by e_{12} and e_{22} or by e_{12} alone. Hence, each non-zero right ideal of R is finitely generated).

3) It can be easily seen that as an Z - module, Z is Noetherian, but not Artinian.

4) The ring $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a \in Q,\, b,\, c \in \Re \right\}$ is right Artinian but not left Artinian.

Since, $\Re$ is a vector space over Q of infinite dimension, there exist real numbers $a_1, a_2, a_3, \ldots, a_n, \ldots$, which are linearly independent over Q. For each positive integer k, $A_k = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in \text{subspace of } \Re \text{ generated by } a_k, a_{k+1}, a_{k+2}, \ldots \right\}$ is a left ideal of R and $A_k > A_{k+1}$ and $a_k \notin A_{k+1}$. Because the set $\{a_1, a_2\ a_3, \ldots\}$ is infinite, we get an infinite strictly descending chain of left ideals and hence R is not left Artinian.

To show that R is right Artinian, let $A_1 > A_2$ be two right ideals of R and $\alpha e_{11} + \beta e_{12} + \gamma e_{22} \in A_2$; where $\alpha \in Q, \beta, \gamma \in \Re$ and $e_{i,j}$ in its usual notation as before. If $\alpha \neq 0$, then either $\alpha e_{11} \in A_2, e_{12}, e_{22} \in A_2$ or $\alpha e_{11} \in A_2, e_{12} \in A_2$ and $\gamma = 0$. In the former situation $(\alpha e_{11})\left(\frac{1}{\alpha}\right)e_{11} \in A_2$. i.e. $e_{11} \in A_2$ and so $A_2 = R$, which is contradiction with $A_1 > A_2$. In the latter situation, A_2 is generated by e_{11} and e_{12}. Now consider $A_1 > A_2 > A_3$; where A_3 is a right ideal of R. Now let $ae_{11} + be_{12} \in A_3$. If $a \neq 0$, then $(ae_{11} + be_{12})\left(\frac{1}{a}\right)e_{11} \in A_3$ and hence $e_{11} \in A_3$; so, $be_{12} \in A_3$. Now if $b \neq 0$ then this implies $be_{12}\left(\frac{1}{b}\right)e_{22} = e_{12} \in A_3$ and thus $A_3 = A_2$, a contradiction. Hence $b = 0$ and A_3 is generated by e_{11}. From this A_3 is a minimal right ideal of R. Hence, the strictly descending chain of right ideals $A_1 > A_2 > A_3 > (0)$ is finite. If $\alpha = 0$ then $\beta e_{12} + \gamma e_{22} \in A_2$. If all other elements of A_2 are of the type $\lambda(\beta e_{12} + \gamma e_{22})$ for some $\lambda \in \Re$, then A_2 is minimal right ideal and $A_1 > A_2 > (0)$ is the strictly descending chain of right ideals. Otherwise, A_2 is generated by either e_{12} and e_{22} or by e_{12} only. In the case of A_2 is generated by e_{12} and e_{22}, if $A_1 > A_2 > A_3$ for some right ideal A_3 of R then either A_3 is generated by e_{12} or by $de_{12} + fe_{22}$ for some $d, f \in \Re$. In each case of A_3 is a minimal right ideal of R and hence R is right Artinian.