Weerakoon-Fernando Method with Accelerated Third Order Convergence for Systems of Nonlinear Equations

HPS Nishani

Department of Mathematics,
Faculty of Applied Sciences,
University of Sri Jayewardenepura,
Gangodawila, Nugegoda, Sri Lanka.
E-mail: hpsnishani@yahoo.com

Sunethra Weerakoon

Department of Mathematics,
Faculty of Applied Sciences,
University of Sri Jayewardenepura,
Gangodawila, Nugegoda, Sri Lanka.
E-mail: sunethra.weerakoon@gmail.com

TGI Fernando

Department of Computer Science,
Faculty of Applied Sciences,
University of Sri Jayewardenepura,
Gangodawila, Nugegoda, Sri Lanka.
E-mail: gishantha@dscs.sjp.ac.lk

Menaka Liyanage

Department of Mathematics,
Faculty of Applied Sciences,
University of Sri Jayewardenepura,
Gangodawila, Nugegoda, Sri Lanka.
E-mail: menakaliyanage@yahoo.com

Abstract Weerakoon-Fernando Method (WFM) is a widely accepted third order iterative method introduced in the late 90s to solve nonlinear equations. Even though it has become so popular among numerical analysts resulting in hundreds of similar work for single variable case, after nearly two decades, nobody took the challenge of extending the method to multivariable systems. In this paper, we extend the WFM to functions of several variables and provide a rigorous proof for the third order convergence. This theory was supported by computational results using several systems of nonlinear equations. Computational algorithms were implemented using MATLAB. We further analyze the method mathematically and demonstrate the reason for the strong performance of WFM computationally, despite it requiring more function evaluations.

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Biographical notes: H.P.S. Nishani completed Bachelors of Science (2012) and Masters of Science in Industrial Mathematics (2015) at the University of Sri Jayewardenepura, Sri Lanka. She currently works as a teacher at Holy Family Convent, Bambalapitaya, Sri Lanka. The research related to this paper was carried out as a part of her thesis work for the Masters degree.

Sunethra Weerakoon retired in 2017 as a Senior Professor of Mathematics from the University of Sri Jayawardenepura, Sri Lanka where she served from 1976. She finished her Bachelors Degree in Mathematics at the University of Peradeniya, Sri Lanka (1972/76). She pursued her Masters & Doctoral Degree studies as a Fulbright Scholar at the Pennsylvania State University, University Park, USA (1979/84). She was a Visiting Teaching Fellow at Curtin University of Technology, Western Australia (1991/93). She was also a Visiting Professor at the Departments of Mathematics, Cornell University (2004/05) and Texas A & M University, College Park (2005/06). She has written a few books: Elementary Numerical Methods, Numerical Solution of Ordinary Differential Equations and Partial Differential Equations and edited a few others. Her research interests are in Numerical Analysis and Partial Differential Equations. She has received a number of awards for her academic achievements and research.

TGI Fernando is presently a senior lecturer in Computer Science in the Department of Computer Science, University of Sri Jayewardenepura, Sri Lanka. He completed his B.Sc. in Mathematics in 1993 and M.Sc. in Industrial Mathematics in 1998 at the University of Sri Jayewardenepura, Sri Lanka. The original Weerakoon-Fernando method with Accelerated Third Order Convergence for Solving Non-linear Equations was developed for his thesis in M.Sc. in Industrial Mathematics, and the second/third-order convergence of the method for single variable non-linear equations were also proved. Subsequently in 2002, he completed M.Sc. in Computer Science at the Asian Institute of Technology (AIT) Thailand. Finally, he completed his Ph.D. in Intelligent Systems at the Brunel University, United Kingdom. He has received several awards for his research achievements. His research interests are mainly in Intelligent Systems, Evolutionary Computing, Swarm Intelligence, Neural Networks (including deep learning neural networks), Machine Learning, Multi-Objective Combinatorial Optimization and Root Finding of Non-linear Equations.

Menaka Liyanage is presently a senior lecturer in the Department of Mathematics, University of Sri Jayewardenepura, Nugegoda, Sri Lanka where she is serving since 1987. She finished her Bachelors degree in Mathematics (1987) at the University of Sri Jayewardenepura and pursued as a Canadian Commonwealth scholar for her Masters degree in Pure Mathematics (1991) at the University of Waterloo, Canada and her doctoral degree (1995) in Mathematics at the McMaster University, Canada. She spent her sabbatical leave in 2004 as a commonwealth fellow at the University of Southampton and the University of Oxford, United Kingdom.

1 Introduction

There are only a limited number of systems of nonlinear equations that can be solved by analytical methods. Thus one has to use iterative numerical methods very frequently when solving systems of nonlinear equations in research and industrial sector.

Among various types of nonlinear equation solvers introduced in the recent past, there is an overwhelming number of research papers with Newton-type formulas and variants of Newton's method. An explosive growth of literature on this topic is visible since the introduction of WFM. Examining over 700 publications citing the introductory paper by Weerakoon and Fernando (2000) alone is sufficient to understand how the research in this area has evolved in the recent past. Peiris et al. (1998) provides an overview of the situation up to the year 2011. Some have tried to formulate methods with higher order convergence paying only a little attention to the required number of function evaluations per iteration and hence jeopardizing the efficiency. Awawdeh (2009); Cordero et al. (2009a); Cordero et al. (2009b); Cordero et al. (2010); Hou and Li (2010); Kim et al. (2009); Mir et al. (2008); Parhi and Gupta (2008); Sharma et al. (2009); Thukral (2010); Wang and Liu (2010); Özban (2000); are a minute fraction of such work trying to achieve the order of convergence as high as 6 or 8.

Among other approaches, some of the researchers who tried to improve Halley's and Chebyshev's methods to obtain more efficient algorithms are Kou and Li (2007a); Kou and Li (2007b); Kou and Li (2007c); Kou (2007a); Kou (2007b); Ezquerro and Hernández (2009). Some tried to get rid of the second derivative requirement. There are algorithms formulated via geometric and various other means. One significant feature among many, even in improving Halley's and Chebyshev methods, they followed the technique used by Weerakoon and Fernando (2000) to prove the order of convergence.

Optimization problems naturally arising in various practical situations require solving systems of nonlinear equations. However, apart from classical methods such as Newton's, Chebyshev's and Halley's, more efficient new algorithms are not available to solve systems of nonlinear equations. Awawdeh (2009) uses a Homotopy analysis method to derive a family of iterative methods to solve systems of nonlinear equations. However, there is a dearth of more efficient system solvers compared to the numerous algorithms available for single-variable case. This is the very reason for us to engage in this research to check whether the WFM that became so popular due to its efficiency would give us similar results when extended for systems.

The Newton's Method is in the forefront in this respect. It uses the vector valued function of several variables and its Jacobian at each iteration. It is known that, under certain conditions, Newton's Method converges to the root quadratically even for functions of several variables Dennis and Robert (1983).

In this paper, we suggest an improvement to the iterations of Newton's Method to solve systems of nonlinear equations by extending the Weerakoon-Fernando Method (WFM) for single variable functions introduced by Fernando (1998); Weerakoon and Fernando (2000). We follow the same improvement and replace the local linear model used in the Newton's method by the superior nonlinear model for WFM and derive the formula to solve systems of nonlinear equations and prove that it also preserves the third order convergence.

Third order convergence of this improvement is verified using some computed results by applying WFM for several variables to a cross section of systems of nonlinear equations. We produce computational results for the proposed method and the Newton's Method using MATLAB. Results are tabulated enabling the comparison of the two iterative methods.

2 Preliminary Results

Definition 2.1. ***Nonlinear functions of several variables***
Let $D \subset \mathbb{R}^n$ and suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a real valued function which assigns a unique real number denoted by $f(x_1, x_2, x_3, ..., x_n)$ to each $(x_1, x_2, x_3, ..., x_n) \in D$. The set D is the domain of f and its range is the set of values that f takes on, that is $\{f(x_1, x_2, x_3, ..., x_n) \in \mathbb{R} | (x_1, x_2, x_3, ..., x_n) \in D\}$. We often write $z = f(x_1, x_2, x_3, ...x_n)$ to make explicit the value taken by f at the general point $(x_1, x_2, x_3, ..., x_n)$. The variables $x_1, x_2, x_3, ..., x_n$ are independent and z is dependent.

Definition 2.2. ***A system of Nonlinear Equations***
A system of nonlinear equations has the following form.

$$\underline{F}(\underline{x}) = \begin{cases} f_1(x_1, x_2, ..., x_n) = 0 \\ f_2(x_1, x_2, ..., x_n) = 0 \\ f_3(x_1, x_2, ..., x_n) = 0 \\ \cdots \\ \cdots \\ f_n(x_1, x_2, ..., x_n) = 0 \end{cases} \tag{2.1}$$

where each f_i is a nonlinear function of n variables.
This system of $n-$ nonlinear equations in n unknowns can alternatively be written as $\underline{F}(\underline{x}) = \underline{0}$ by defining a vector valued function $\underline{F}(\underline{x})$.
Note: The first derivative and the second derivative of $\underline{F}(\underline{x})$, when they exist, will be the Jacobian matrix $J(\underline{F}(\underline{x}))$ and the array of Hessian matrices $H(\underline{F}(\underline{x}))$, respectively.

Definition 2.3. ***Order of Convergence of an iterative scheme***
Let the iterative sequence $\{\underline{x}_n\}; n = 1, 2, ...$ that converges to $\underline{x}^*$ be generated by a numerical scheme. If there exists a constant $C \geq 0$, an integer $n_0 \geq 0$ and $\rho \geq 0$ such that for all $n > n_0$, the inequality (2.2) holds for any vector norm $\|.\|$,

$$\left\|\underline{x}_{n+1} - \underline{x}^*\right\| \leq C \|\underline{x}_n - \underline{x}^*\|^\rho \tag{2.2}$$

then the iterative scheme is said to converge to $\underline{x}^*$ with ρ^{th} order convergence.

Definition 2.4. ***Computational Order of Convergence***
Let $\underline{x}^*$ be a root of the equation $\underline{F}(\underline{x}) = \underline{0}$ and suppose that $\underline{x}_{n-1}$, $\underline{x}_n$ and $\underline{x}_{n+1}$ be consecutive iterations closer to the root $\underline{x}^*$, generated by an iterative scheme. Then the Computational Order of Convergence (COC) ρ of the iterative scheme or the numerical algorithm can be approximated by:

$$\rho = \frac{ln[\frac{\left\|\underline{x}_{n+1} - \underline{x}^*\right\|}{\left\|\underline{x}_n - \underline{x}^*\right\|}]}{ln[\frac{\left\|\underline{x}_n - \underline{x}^*\right\|}{\left\|\underline{x}_{n-1} - \underline{x}^*\right\|}]} \tag{2.3}$$

Theorem 2.1. ***(Existence of Matrix inverse)*** Let $A \in \mathbb{R}^n$ be a square matrix and $\rho(A)$ be the spectral radius of A, then $(I_n - A)^{-1}$ exists and

$$(I_n - A)^{-1} = \sum_{k=0}^{\infty} A^k \tag{2.4}$$

if and only if $\rho(A) < 1$.

2.1 Newton's Method to solve systems of nonlinear equations

Newton's method is an iterative method to approximate a single root $\underline{x}^*$ of the system of nonlinear equations $\underline{F}(\underline{x}) = \underline{0}$. The process starts with an initial approximation $\underline{x}_{(0)}$ which is closer to $\underline{x}^*$. Its iterative formula is given by;

$$\underline{x}_{(i+1)} = \underline{x}_{(i)} - [J(\underline{F}(\underline{x}_{(i)}))]^{-1}\underline{F}(\underline{x}_{(i)}); i = 0, 1, ... \tag{2.5}$$

when $[J(\underline{F}(\underline{x}_{(i)}))]^{-1}$ exists. Here $\underline{x}_{(i)}$ is the ith iterate. It is well known that Newton's method is quadratically convergent.

3 Weerakoon-Fernando Method (WFM) to solve nonlinear equations in one variable

Weerakoon and Fernando (2000) introduced a third order convergent Weerakoon - Fernando Method (WFM) to solve nonlinear equations. The local model $M_n(x)$ of WFM is given by the equation (3.1).

$$M_n(x) = f(x_n) + \frac{1}{2}(x - x_n)[f^{'}(x_n) + f^{'}(x)] \tag{3.1}$$

This leads to the implicit scheme of the Weerakoon-Fernando Method given by,

$$\begin{aligned} &x_{n+1} = x_n - \tfrac{2f(x_n)}{[f^{'}(x_n)+f^{'}(x^*_{n+1})]} \\ &where; \\ &x^*_{n+1} = x_n - \tfrac{f(x_n)}{f^{'}(x_n)}; \qquad n = 0, 1, 2, ... \end{aligned} \tag{3.2}$$

4 Derivation of the Weerakoon-Fernando Method (WFM) to solve systems of nonlinear equations

Lemma 4.1. *When $\underline{F}$ is a vector valued function with non zero derivatives defined on the set $D \subset \mathbb{R}^n$ and $\underline{x}, \underline{x}_0 \in D$, the extension of the WFM to solve systems of nonlinear equations can be given as follows.*

$$\begin{aligned} &\underline{x}_{n+1} = \underline{x}_n - 2[J(\underline{F}(\underline{x}_n)) + J(\underline{F}(\underline{x}^{\lambda}{}_{n+1}))]^{-1}\underline{F}(\underline{x}_n) \\ &where; \\ &\underline{x}^{\lambda}_{n+1} = \underline{x}_n - [J(\underline{F}(\underline{x}_n))]^{-1}\underline{F}(\underline{x}_n); \qquad n = 0, 1, 2, ... \end{aligned} \tag{4.1}$$

Proof: *Taylor's expansion of $\underline{F}(\underline{x})$ as given in Ortega and Rheinboldt (1970) is:*

$$\begin{aligned} \underline{F}(\underline{x}) = {} & \underline{F}(\underline{x}_n) + \underline{F}^{'}(\underline{x}_n)(\underline{x} - \underline{x}_n) + \tfrac{1}{2}\underline{F}^{(2)}(\underline{x}_n)(\underline{x} - \underline{x}_n)^2 + ... \\ & + \tfrac{1}{(k-1)!}\underline{F}^{(k-1)}(\underline{x}_n)(\underline{x} - \underline{x}_n)^{k-1} + \int_0^1 \tfrac{(1-t)^{k-1}}{(k-1)!}\underline{F}^{(k)}(\underline{x}_n + t(\underline{x} - \underline{x}_n))(\underline{x} - \underline{x}_n)^k dt \end{aligned}$$

For $k = 1$;

$$\underline{F}(\underline{x}) = \underline{F}(\underline{x}_n) + \int_0^1 J(\underline{F}(\underline{x}_n + t(\underline{x} - \underline{x}_n))(\underline{x} - \underline{x}_n))dt \tag{4.2}$$

Let $\underline{\lambda} = \underline{x}_n + t(\underline{x} - \underline{x}_n)$. *Then:*

$$d\underline{\lambda} = (\underline{x} - \underline{x}_n)dt \tag{4.3}$$

Then:

$$\int_0^1 J(\underline{F}(\underline{x}_n + t(\underline{x} - \underline{x}_n))(\underline{x} - \underline{x}_n))dt = \int_{\underline{x}_n}^{\underline{x}} J(\underline{F}(\underline{\lambda}))d\underline{\lambda} \tag{4.4}$$

The indefinite integral of equation (4.4) *is approximated as follows,*

$$\int_{\underline{x}}^{\underline{x}_n} J(\underline{F}(\underline{\lambda}))d\underline{\lambda} \approx \frac{1}{2}[J(\underline{F}(\underline{x}_n)) + J(\underline{F}(\underline{x}))](\underline{x} - \underline{x}_n) \tag{4.5}$$

Then the equation (4.2) *becomes*

$$\underline{F}(\underline{x}) \approx \underline{F}(\underline{x}_n) + \frac{1}{2}[J(\underline{F}(\underline{x}_n)) + J(\underline{F}(\underline{x}))](\underline{x} - \underline{x}_n)$$

Thus we propose local model $\underline{M}_n(\underline{x})$ *to approximate* $\underline{F}(\underline{x})$ *as follows.*

$$\underline{M}_n(\underline{x}) = \underline{F}(\underline{x}_n) + \frac{1}{2}[J(\underline{F}(\underline{x}_n)) + J(\underline{F}(\underline{x}))](\underline{x} - \underline{x}_n) \tag{4.6}$$

If $\underline{x} = \underline{x}_{n+1}$ *is the root, then*

$$\underline{M}_n(\underline{x}) = \underline{M}_n(\underline{x}_{n+1}) = \underline{0}$$

Thus equation (4.6) *gives:*

$$\begin{aligned} &\underline{M}_n(\underline{x}_{n+1}) = \underline{F}(\underline{x}_n) + \tfrac{1}{2}[J(\underline{F}(\underline{x}_n)) + J(\underline{F}(\underline{x}_{n+1}))](\underline{x}_{n+1} - \underline{x}_n) = \underline{0} \\ &\Rightarrow \quad x_{n+1} = x_n - 2[J(\underline{F}(\underline{x}_n)) + J(\underline{F}(\underline{x}_{n+1}))]^{-1}\underline{F}(\underline{x}_n) \end{aligned} \tag{4.7}$$

This is an implicit scheme because it requires $(n+1)^{th}$ *iterative step to find* $J(\underline{F}(\underline{x}_{n+1}))$. *As in the case of the one variable, we use Newton's iterative step to compute* $\underline{x}_{n+1}$ *in the RHS as follows.*

$$\underline{x}^{\lambda}_{n+1} = \underline{x}_n - [J(\underline{F}(\underline{x}_n))]^{-1}\underline{F}(\underline{x}_n); \qquad n = 0, 1, 2, ...$$

Hence the Lemma 4.1.

5 Establishment of the third order convergence of WFM

Theorem 5.1. ***(Third Order Convergence)***
Let $\underline{F} : D \to \mathbb{R}^n$ be a twice continuously differentiable vector valued function in the open convex set $D \subset \mathbb{R}^n$. Assume that there exists (i) $\underline{x}^* \in D$ s.t. $\underline{F}(\underline{x}^*) = \underline{0}$, (ii) $\sigma > 0$ s.t. $\|J(\underline{F}(\underline{x}))\| > \sigma$ for every $\underline{x}$ in the neighbourhood $N(\underline{x}^*, \underline{r})$ of $\underline{x}^*$ and (iii) the inverse of $J(\underline{F}(\underline{x}))$ for all $\underline{x} \in D$, then

the order of convergence of the sequence generated by the WFM for systems of nonlinear equations given by

$$\begin{aligned} &\underline{x}_{n+1} = \underline{x}_n - 2[J(\underline{F}(\underline{x}_n)) + J(\underline{F}(\underline{x}^{\lambda}{}_{n+1}))]^{-1}\underline{F}(\underline{x}_n) \\ &where; \\ &\underline{x}^{\lambda}_{n+1} = \underline{x}_n - [J(\underline{F}(\underline{x}_n))]^{-1}\underline{F}(\underline{x}_n); \qquad n = 0, 1, 2, ... \end{aligned} \tag{5.1}$$

satisfies the following equation demonstrating the third order convergence.

$$\left\|\underline{E}_{n+1}\right\| \le \left\|C_2{}^2 + \frac{1}{2}C_3\right\| \|\underline{E}_n\|^3 \tag{5.2}$$

Here $\underline{E}_n = \underline{x}_n - \underline{x}^*$ and $C_k = \frac{1}{k!}[J(\underline{F}(\underline{x}^*))]^{-1}\underline{F}^{(k)}(\underline{x}^*)$

Proof: By Taylor's Expansion Ortega and Rheinboldt (1970) for $\underline{F}(\underline{x})$ we have:

$$\underline{F}(\underline{x}) = \underline{F}(\underline{x}^*) + J(\underline{F}(\underline{x}^*))(\underline{x} - \underline{x}^*) + \frac{1}{2!}\underline{F}^{(2)}(\underline{x}^*)(\underline{x} - \underline{x}^*)^2 + \frac{1}{3!}\underline{F}^{(3)}(\underline{x}^*)(\underline{x} - \underline{x}^*)^3 + ...$$

Substituting $\underline{F}(\underline{x}^*) = \underline{0}$, since $\underline{x}^*$ is the root, we have

$$\begin{aligned} \underline{F}(\underline{x}) &= J(\underline{F}(\underline{x}^*))(\underline{x} - \underline{x}^*) + \tfrac{1}{2!}\underline{F}^{(2)}(\underline{x}^*)(\underline{x} - \underline{x}^*)^2 + \tfrac{1}{3!}\underline{F}^{(3)}(\underline{x}^*)(\underline{x} - \underline{x}^*)^3 + ... \\ &= \underline{J}(\underline{F}(\underline{x}^*)[(\underline{x} - \underline{x}^*) + \tfrac{1}{2!}[J(\underline{F}(\underline{x}^*))]^{-1}\underline{F}^{(2)}(\underline{x}^*)(\underline{x} - \underline{x}^*)^2 \\ &\quad + \tfrac{1}{3!}[J(\underline{F}(\underline{x}))]^{-1}\underline{F}^{(3)}(\underline{x}^*)(\underline{x} - \underline{x}^*)^3 + ...] \end{aligned} \tag{5.3}$$

when $\underline{x} = \underline{x}_n$:

$$\begin{aligned} \underline{F}(\underline{x}_n) &= \underline{J}(\underline{F}(\underline{x}^*)\{(\underline{x}_n - \underline{x}^*) + \tfrac{1}{2!}[J(\underline{F}(\underline{x}^*))]^{-1}\underline{F}^{(2)}(\underline{x}^*)(\underline{x}_n - \underline{x}^*)^2 \\ &\quad + \tfrac{1}{3!}[J(\underline{F}(\underline{x}^*))]^{-1}\underline{F}^{(3)}(\underline{x}^*)(\underline{x}_n - \underline{x}^*)^3 + O((\underline{x}_n - \underline{x}^*)^4)\} \\ \underline{F}(\underline{x}_n) &= J(\underline{F}(\underline{x}^*))[\underline{E}_n + C_2(\underline{E}_n)^2 + C_3(\underline{E}_n)^3 + O((\underline{E}_n)^4)] \end{aligned} \tag{5.4}$$

where, $\qquad C_k = \frac{1}{k!}[J(\underline{F}(\underline{x}^*))]^{-1}\underline{F}^{(k)}(\underline{x}^*)$, $k = 1, 2, ...$

Differentiating equation (5.3) with respect to $\underline{x}$:

$$J(\underline{F}(\underline{x})) = J(\underline{F}(\underline{x}^*)) + \frac{2}{2!}\underline{F}^{(2)}(\underline{x}^*)(\underline{x} - \underline{x}^*) + \frac{3}{3!}\underline{F}^{(3)}(\underline{x}^*)(\underline{x} - \underline{x}^*)^2 + ... \tag{5.5}$$

when $\underline{x} = \underline{x}_n$:

$$\begin{aligned} J(\underline{F}(\underline{x}_n)) &= J(\underline{F}(\underline{x}^*)) + \tfrac{2}{2!}\underline{F}^{(2)}(\underline{x}^*)(\underline{x}_n - \underline{x}^*) + \tfrac{3}{3!}\underline{F}^{(3)}(\underline{x}^*)(\underline{x}_n - \underline{x}^*)^2 + ... \\ &= J(\underline{F}(\underline{x}^*)) + \tfrac{2}{2!}\underline{F}^{(2)}(\underline{x}^*)(\underline{E}_n) + \tfrac{3}{3!}\underline{F}^{(3)}(\underline{x}^*)(\underline{E}_n)^2 + ... \\ &= J(\underline{F}(\underline{x}^*))[I + \tfrac{2}{2!}[J(\underline{F}(\underline{x}^*))]^{-1}\underline{F}^{(2)}(\underline{x}^*)(\underline{E}_n) + \tfrac{3}{3!}[J(\underline{F}(\underline{x}^*))]^{-1}\underline{F}^{(3)}(\underline{x}^*)(\underline{E}_n)^2 \\ &\quad + O((\underline{E}_n)^3)] \\ J(\underline{F}(\underline{x}_n)) &= J(\underline{F}(\underline{x}^*))[I + 2C_2(\underline{E}_n) + 3C_3(\underline{E}_n)^2] + O((\underline{E}_n)^3) \end{aligned} \tag{5.6}$$

Then $[J(\underline{F}(\underline{x}^*))]^{-1}$ is expressed as:

$$[J(\underline{F}(\underline{x}_n))]^{-1} = [J(\underline{F}(\underline{x}^*))[I + 2C_2(\underline{E}_n) + 3C_3(\underline{E}_n)^2] + O((\underline{E}_n)^3)]^{-1} \tag{5.7}$$

Applying inversion of matrices equation (2.4) to equation (5.7):

$$\begin{aligned}[J(\underline{F}(\underline{x}_n))]^{-1} &= [I-(2C_2(\underline{E}_n)+3C_3(\underline{E}_n)^2)+(2C_2(\underline{E}_n)+3C_3(\underline{E}_n)^2)^2 \\ &+O((\underline{E}_n)^3)][J(\underline{F}(\underline{x}^*))]^{-1}\end{aligned} \tag{5.8}$$

Then multiplying equation (5.8) with equation (5.4) we get

$$\begin{aligned}[J(\underline{F}(\underline{x}_n))]^{-1}\underline{F}(\underline{x}_n) &= [I-(2C_2(\underline{E}_n)+3C_3(\underline{E}_n)^2)+(2C_2(\underline{E}_n)+3C_3(\underline{E}_n)^2)^2 \\ &+O((\underline{E}_n)^3)][J(\underline{F}(\underline{x}^*))]^{-1}\times J(\underline{F}(\underline{x}^*))[\underline{E}_n+C_2(\underline{E}_n)^2+C_3(\underline{E}_n)^3+O((\underline{E}_n)^4)] \\ &= [I-(2C_2(\underline{E}_n)+3C_3(\underline{E}_n)^2)+(2C_2(\underline{E}_n)+3C_3(\underline{E}_n)^2)^2+O((\underline{E}_n)^3)] \\ &\times[\underline{E}_n+C_2(\underline{E}_n)^2+C_3(\underline{E}_n)^3+O((\underline{E}_n)^4)] \\ &= \underline{E}_n+C_2(\underline{E}_n)^2+C_3(\underline{E}_n)^3-2C_2(\underline{E}_n)^2-2{C_2}^2(\underline{E}_n)^3-3C_3(\underline{E}_n)^3 \\ &+4{C_2}^2(\underline{E}_n)^3+O({\underline{E}_n}^4)\end{aligned}$$

$$[J(\underline{F}(\underline{x}_n))]^{-1}\underline{F}(\underline{x}_n) = \underline{E}_n - C_2(\underline{E}_n)^2 + 2({C_2}^2 - C_3)(\underline{E}_n)^3 + O({\underline{E}_n}^4) \tag{5.9}$$

Substituting equation (5.9) to equation (4.1) we get

$$\begin{aligned}\underline{x}^{\lambda}_{n+1} &= \underline{x}_n - [\underline{E}_n - C_2(\underline{E}_n)^2 + 2({C_2}^2 - C_3)(\underline{E}_n)^3 + O({\underline{E}_n}^4)] \\ &= (\underline{x}^* + \underline{E}_n) - [\underline{E}_n - C_2(\underline{E}_n)^2 + 2({C_2}^2 - C_3)(\underline{E}_n)^3 + O({\underline{E}_n}^4)]\end{aligned}$$

$$\Longrightarrow \underline{x}^{\lambda}_{n+1} = \underline{x}^* + C_2(\underline{E}_n)^2 + 2(C_3 - {C_2}^2)(\underline{E}_n)^3 + O({\underline{E}_n}^4) \tag{5.10}$$

Substituting $\underline{x}^{\lambda}_{n+1}$ of (5.10) to $\underline{x}$ in (5.5) we get the following.

$$\begin{aligned}J(\underline{F}(\underline{x}^{\lambda}_{n+1})) &= J(\underline{F}(\underline{x}^*)) + \tfrac{2}{2!}\underline{F}^{(2)}(\underline{x}^*)(\underline{x}^* + C_2(\underline{E}_n)^2 + 2(C_3 - {C_2}^2)(\underline{E}_n)^3 + O({\underline{E}_n}^4) - \underline{x}^*) \\ &\tfrac{3}{3!}\underline{F}^{(3)}(\underline{x}^*)(\underline{x}^* + C_2(\underline{E}_n)^2 + 2(C_3 - {C_2}^2)(\underline{E}_n)^3 + O({\underline{E}_n}^4) - \underline{x}^*)^2 + ... \\ &= J(\underline{F}(\underline{x}^*))[I + 2C_2(C_2(\underline{E}_n)^2 + 2(C_3 - {C_2}^2)(\underline{E}_n)^3 + O({\underline{E}_n}^4) \\ &+3C_3(C_2(\underline{E}_n)^2 + 2(C_3 - {C_2}^2)(\underline{E}_n)^3 + O({\underline{E}_n}^4))^2] \\ &= J(\underline{F}(\underline{x}^*))[I + 2C_2^2(\underline{E}_n)^2 + 4C_2(C_3 - C_2^2)(\underline{E}_n)^3 + O(\underline{E}_n^4)]\end{aligned} \tag{5.11}$$

Adding equations (5.6) and (5.11)

$$\begin{aligned}[J(\underline{F}(\underline{x}_n)) + J(\underline{F}(\underline{x}^{\lambda}_{n+1}))] &= J(\underline{F}(\underline{x}^*))[I + 2C_2(\underline{E}_n) + 3C_3(\underline{E}_n)^2] + O((\underline{E}_n)^3) \\ &+J(\underline{F}(\underline{x}^*))[I + 2C_2^2(\underline{E}_n)^2 + 4C_2(C_3 - C_2^2)(\underline{E}_n)^3 + O(\underline{E}_n^4)] \\ &= 2\times J(\underline{F}(\underline{x}^*))[I + C_2(\underline{E}_n) + (C_2^2 + \tfrac{3}{2}C_3)(\underline{E}_n)^2 + O(\underline{E}_n^3)]\end{aligned}$$

Then $[J(\underline{F}(\underline{x}_n)) + J(\underline{F}(\underline{x}^{\lambda}_{n+1}))]^{-1}$ can be expressed as:

$$\begin{aligned}[J(\underline{F}(\underline{x}_n)) + J(\underline{F}(\underline{x}^{\lambda}_{n+1}))]^{-1} &= 2\times\{J(\underline{F}(\underline{x}^*))[I + C_2(\underline{E}_n) + (C_2^2 + \tfrac{3}{2}C_3)(\underline{E}_n)^2 + O(\underline{E}_n^3)]\}^{-1} \\ &= \tfrac{1}{2}[I + C_2(\underline{E}_n) + (C_2^2 + \tfrac{3}{2}C_3)(\underline{E}_n)^2 + O(\underline{E}_n^3)]^{-1}\times[J(\underline{F}(\underline{x}^*))]^{-1}\end{aligned} \tag{5.12}$$

Applying inversion of matrices equation (2.4) to equation (5.12):

$$\begin{aligned}&[J(\underline{F}(\underline{x}_n)) + J(\underline{F}(\underline{x}^{\lambda}_{n+1}))]^{-1} \\ &= \tfrac{1}{2}[I - C_2(\underline{E}_n) - C_2^2(\underline{E}_n)^2 - (\tfrac{3}{2}C_3)(\underline{E}_n)^2 + (C_2(\underline{E}_n))^2 + O(\underline{E}_n^3)]\times[J(\underline{F}(\underline{x}^*))]^{-1} \\ &= \tfrac{1}{2}[I - C_2(\underline{E}_n) - (\tfrac{3}{2}C_3)(\underline{E}_n)^2 + O(\underline{E}_n^3)]\times[J(\underline{F}(\underline{x}^*))]^{-1}\end{aligned} \tag{5.13}$$

Finally, substituting equations (5.13) and (5.4) to equation (4.1) with
$\underline{x}_n = \underline{x}^* + \underline{E}_n$ and $\underline{x}_{n+1} = \underline{x}^* + \underline{E}_{n+1}$

$$\begin{aligned}\underline{x}^* + \underline{E}_{n+1} = \underline{x}^* + \underline{E}_n - 2 \times \tfrac{1}{2}[I - C_2(\underline{E}_n) - (\tfrac{3}{2}C_3)(\underline{E}_n)^2 + O(\underline{E}_n^3)] \times [J(\underline{F}(\underline{x}^*))]^{-1} \\ \times J(\underline{F}(\underline{x}^*))[\underline{E}_n + C_2(\underline{E}_n)^2 + C_3(\underline{E}_n)^3 + O((\underline{E}_n)^4)]\end{aligned}$$

$$\underline{E}_{n+1} = (C_2 + \frac{1}{2}C_3)(\underline{E}_n)^3 + O(\underline{E}_n^4)$$

$$\|\underline{E}_{n+1}\| \leq \left\|(C_2^2 + \frac{1}{2}C_3)\right\| \|\underline{E}_n\|^3$$

Hence the third order convergence of the Weerakoon-Fernando Method (WFM) to solve systems of nonlinear equations, by (2.2).

6 Computational Results

Generated computational results for a cross section of systems of nonlinear equations are given in Tables 6.1, 6.2, 6.3, 6.4 and 6.5.

7 Discussion and Conclusion

Computational results given in Tables 6.1, 6.2, 6.3, 6.4 and 6.5 overwhelmingly support the theory that WFM is third order convergent. Apparently, the WFM needs one more Jacobian evaluation at each iteration when compared with the Newton's method. However, it is evident by the computed results presented in Tables 6.1, 6.2, 6.3, 6.4 and 6.5 that the total number of Jacobian evaluations required by the WFM is less or almost the same. Further, as shown by almost all examples presented, the WFM seems to be behaving very favorably for systems having trigonometric and exponential functions.

In the quest for an explanation to the extraordinary performance of the WFM, we realized that the nonlinear local model $\underline{M}_n(\underline{x})$:

$$\underline{M}_n(\underline{x}) = \underline{F}(\underline{x}_n) + \frac{1}{2}[J(\underline{F}(\underline{x})) + J(\underline{F}(\underline{x}_n))](\underline{x} - \underline{x}_n) \tag{7.1}$$

proposed in the process of deriving WFM possesses very special qualities embedded in it. At $\underline{x} = \underline{x}_n$;

$$\begin{aligned}\underline{M}_n(\underline{x}_n) &= \underline{F}(\underline{x}_n) + \tfrac{1}{2}[J(\underline{F}(\underline{x}_n)) + J(\underline{F}(\underline{x}_n))](\underline{x}_n - \underline{x}_n) \\ &= \underline{F}(\underline{x}_n)\end{aligned} \tag{7.2}$$

Hence, the local model of WFM to solve functions of several variables agrees with the function $\underline{F}(\underline{x})$, when $\underline{x} = \underline{x}_n$.

Differentiating the equation (7.1) with respect to $\underline{x}$ gives:

$$\underline{M}_n^{'}(\underline{x}) = \frac{1}{2}[J(\underline{F}(\underline{x})) + J(\underline{F}(\underline{x}_n)) + J^{'}(\underline{F}(\underline{x}))(\underline{x} - \underline{x}_n)] \tag{7.3}$$

At $\underline{x} = \underline{x}_n$;

$$\begin{aligned} \underline{M}_n^{'}(\underline{x}_n) &= \tfrac{1}{2}[J(\underline{F}(\underline{x}_n)) + J(\underline{F}(\underline{x}_n)) + J^{'}(\underline{F}(\underline{x}_n))(\underline{x}_n - \underline{x}_n)] \\ &= \tfrac{1}{2}[2J(\underline{F}(\underline{x}_n))] \\ &= J(\underline{F}(\underline{x}_n)) \end{aligned} \tag{7.4}$$

We know that $\underline{F}^{'}(\underline{x}) = J(\underline{F}(\underline{x}))$, where $J(\underline{F}(\underline{x}))$ is the Jacobian of $\underline{F}(\underline{x})$.
Hence, the first derivative of the local model of WFM to solve functions of several variables agrees with the first derivative $J(\underline{F}(\underline{x}))$ of the function, when $\underline{x} = \underline{x}_n$.

Differentiating the equation (7.3) with respect to $\underline{x}$,

$$\underline{M}_n^{(2)}(\underline{x}) = \frac{1}{2}[H(\underline{F}(\underline{x})) + H(\underline{F}(\underline{x}_n)) + H^{'}(\underline{F}(\underline{x}))(\underline{x} - \underline{x}_n)] \tag{7.5}$$

At $\underline{x} = \underline{x}_n$;

$$\begin{aligned} \underline{M}_n^{(2)}(\underline{x}_n) &= \tfrac{1}{2}[H(\underline{F}(\underline{x}_n)) + H(\underline{F}(\underline{x}_n)) + H^{'}(\underline{F}(\underline{x}))(\underline{x}_n - \underline{x}_n)] \\ &= \tfrac{1}{2}[2H(\underline{F}(\underline{x}_n))] \\ &= H(\underline{F}(\underline{x}_n)) \end{aligned} \tag{7.6}$$

Here, $H(\underline{F}(\underline{x})) = J^{'}(\underline{F}(\underline{x})$ is the array of Hessian matrices representing the derivative of the Jacobian or the second derivative of $\underline{F}(\underline{x})$.

Hence, the second derivative of the local model of WFM to solve functions of several variables agrees with the second derivative $H(\underline{F}(\underline{x}))$ of the function, when $\underline{x} = \underline{x}_n$.

As the equations (7.2), (7.4) and (7.6) demonstrate not only $\underline{M}_n(\underline{x}_n)$ and its derivative agrees with the function and its derivative but also its second derivative agrees with the second derivative of the function, at each iterative point.

This is the reason for the efficiency and the third order convergence demonstrated by the WFM.

Now that we have provided a rigorous proof of the third order convergence of the WFM for systems of nonlinear equations and supported the theory with very strong computational evidence, research community and the industry can apply this efficient algorithm as a credible nonlinear system solver without hesitation.

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Table 6.1: Computed Results for the functions of two variables

Functions	Initial Guess	No.of iterations		COC		Root
$\underline{F}(\underline{X})$	$\underline{X}^{(0)}$	NM	WFM	NM	WFM	
$x_1^4 + x_2^4 - 67$ $x_1^3 - 3x_1x_2^2 + 35$	$(10, 20)$	16	11	1.992	2.938	$(1.88364520891082, 2.71594753880345)$
	$(2, 3)$	8	5	1.9895	2.9265	$-Do-$
$x_1^2 - 10x_1 + x_2^2 + 8$ $x_1x_2^2 + x_1 - 10x_2 + 8$	$(0, 0)$	5	2	1.9919	ND	$(1, 1)$
	$(1-, -2)$	6	4	1.946	3.0203	$-Do-$
	$(5, -2)$	107	17	2.02	2.831	$-Do-$
$-x_1^2 - x_1 + 2x_2 - 18$ $(x_1 - 1)^2 + (x_2 - 6)^2 - 25$	$(-5, 5)$	9	6	1.9967	2.8285	$(-2, 10)$
$2cos(x_2) + 7sin(x_1) - 10x_1$ $7cos(x_1) - 2sin(x_2) - 10x_2$	$(10, 10)$	9	7	1.943	2.761	$(0.526522621918048, 0.507919719037091)$
	$(1, 1)$	5	2	2.006	ND	$-Do-$
$x_1 - cos(x_2)$ $sin(x_1) + 0.5x_2$	$(0.785, 0.785)$	$more$	6	$diverge$	3.51	$(0.5303886895, -1.011737334)$
$x_1^2 + x_2^2 - 2$ $e^{x_1 - 1} + x_2^3 - 2$	$(2, 3)$	8	5	1.9888	3.6677	$(1, 1)$

COC-Computational Order of Convergence NM-Newton's Method Tolerance = 1E-15
WFM-Weerakoon-Fernando Method ND-Not Defined as the No. of iterations ≤ 3

Table 6.2: Computed Results for the functions of three variables

Functions $\underline{F}(\underline{X})$	Initial Guess $\underline{X}^{(0)}$	No.of iterations NM	No.of iterations WFM	COC NM	COC WFM	Root
$x_1^2+x_2^2+x_3^2$ $x_1^2-x_2^2+x_3^2$ $x_1^2+x_2^2-x_3^2$	$(1,1,1)$	51	32	2.25	3.47	$(0,0,0)$
$3x_1^2-cos(x_2x_3)-\frac{1}{2}$ $x_1^2-81(x_2+0.1)^2+sin(x_3)+1.06$ $e^{-x_1x_2}+20x_3+\frac{10\pi-3}{3}$	$(0.1,0.1,-0.1)$	8	5	1.9648	2.8165	$(0.5,0,-0.52359877)$
$x_1+e^{x_1-1}+(x_2+x_3)^2-27$ $\frac{e^{x_2-2}}{x_1}+x_3^2-10$ $x_2^2+sin(x_2-2)+x_3-7$	$(4,4,4)$	15	8	2.0825	3.1063	$(1,2,3)$

COC-Computational Order of Convergence NM-Newton's Method
WFM-Weerakoon-Fernando Method Tolerance = 1E-15

Table 6.3: Computed Results for the functions of four variables: $f_1 = x_1 + x_2 - 2, f_2 = x_1x_3 + x_2x_4, f_3 = x_1x_3^2 + x_2x_4^2 - 2/3, f_4 = x_1x_3^3 + x_2x_4^3$ Ariyaratne et al. (2017)

Initial Guess $\underline{X}^{(0)}$	No of iterations		COC		Root
	NM	WFM	NM	WFM	
$(10, 10, 2, -1)$	8	6	1.789	3.652	$(1.000000000000000, 1.000000000000000, 0.577350269189626, -0.577350269189626)$
$(9.449645849092210, 8.198130316168244, 1.958279956709287, -2.229958415751701)$	8	6	2.187	3.086	-Do-
$(10, 10, -1, 2)$	8	6	1.789	3.652	$(1.000000000000000, 1.000000000000000, -0.577350269189626, 0.577350269189626)$
$(10.143204223004453, 9.780732661184169 -2.504270889607720, 1.961429173872073)$	8	6	2.487	3.015	-Do-
$(8.422837706326886, 8.568164487615993, -2.334158236494317, 2.483834575741233)$	8	6	1.798	2.995	-Do-

COC-Computational Order of Convergence   NM-Newton's Method
WFM-Weerakoon-Fernando Method   Tolerance = 1E-15

Table 6.4: Computed Results for the functions of six variables: $f_1 = x_1^2 + x_3^2 - 1, f_2 = x_2^2 + x_4^2 - 1, f_3 = x_5 x_3^3 + x_6 x_4^3, f_4 = x_5 x_1^3 + x_6 x_2^3, f_5 = x_5 x_1 x_3^2 + x_6 x_4^2 x_2, f_6 = x_5 x_1^2 x_3 + x_6 x_2^2 x_4$ Ariyaratne et al. (2017)

Initial Guess $\underline{X}^{(0)}$	No of iterations		COC		Root
	NM	WFM	NM	WFM	
$(3.5, 4.6, 5.5, 2, 1, -4)$	7	5	2.277	3.780	$(0.536875492193160, 0.917070056253201, .843661487732107, 0.398726111414529, 0.000000000000000, -0.000000000000000)$
$(2.254647907165964, 3.749992692264525, 3.864453854746140, 2.304837522652028, 1.733452973178612, -3.813627541638709)$	7	5	2.321	4.275	$(0.503934920994756, 0.851947220856559, 0.863741625373010, 0.523627666261807, -0.000000000000000, 0.000000000000000)$
$(2.525763968916589, 5.053842947251501, 5.828996658108908, 2.162957348497639, 2.479764129432029, -4.940883894097480)$	7	5	2.420	4.454	$(0.397589601576613, 0.919340883394728, 0.917563354062350, 0.393461993233146 -0.000000000000000, 0.000000000000000)$
$(3.589981223108409, 6.574818496483408, 4.374706529598536, 0.423193093000913, -0.561210141907223, -5.745634516099577)$	7	5	2.603	5.701	$(0.634367135513758, 0.997934939782339, 0.773031912264992, 0.064232826199848 -0.000000000000000, 0.000000000000000)$
$(2.471141431283848, 4.369609252007773, 6.251184340480428, 1.436912841607442, 1.945360297204807, -4.421170098884947)$	7	5	2.625	4.817	$(0.367625820251306, 0.949955380939287, 0.929973793332132, 0.312385617825939 0.000000000000000, 0.000000000000000)$

COC-Computational Order of Convergence   NM-Newton's Method
WFM-Weerakoon-Fernando Method   Tolerance = 1E-15

Table 6.5: Computed Results for the functions of ten variables: $f_1 = x_1 - 0.25428722 - 0.18324757x_4x_3x_9, f_2 = x_2 - 0.37842197 - 0.16275449x_1x_{10}x_6, f_3 = x_3 - 0.27162577 - 0.16955071x_1x_2x_{10}, f_4 = x_4 - 0.19807914 - 0.15585316x_7x_1x_6, f_5 = x_5 - 0.44166728 - 0.19950920x_7x_6x_3, f_6 = x_6 - 0.14654113 - 0.18922793x_8x_5x_{10}, f_7 = x_7 - 0.42937161 - 0.21180486x_2x_5x_8, f_8 = x_8 - 0.07056438 - 0.17081208x_1x_7x_6, f_9 = x_9 - 0.34504906 - 0.19612740x_10x_6x_8, f_{10} = x_{10} - 0.42651102 - 0.21466544x_4x_8x_1$ Ariyaratne et al. (2017)

Initial Guess $\underline{X}^{(0)}$	No of iterations		COC		Root
	NM	WFM	NM	WFM	
$(1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$	5	3	1.799	3.701	$(0.257833393700504, 0.381097154602807, 0.278745017346440, 0.200668964225344, 0.445251424841042, 0.149183919969355, 0.432009698983720, 0.073402777776249, 0.345966826875554, 0.427326275993290)$
$(-3.095667328200459, -1.310834539361051, -0.392740627395884, 4.816379509707497, -3.435950477734365, 3.555228058459113, 1.447645368700879, -1.237277897211684, -3.090763047636970, -0.717470070206140)$	10	6	1.827	3.043	-Do-
$(4.243762666888351, 5.826633997219503, 3.302487922675977, -0.561229958850169, 1.840693332784520, -2.922309847562570, 5.063081506497330, 4.796537244819048, 4.177605593706419, -1.392720009445354)$	9	6	1.926	3.501	-Do-
$(2.959493133016079, 2.998878499282916, 2.385307582718379, -3.663961639335705, -3.311939008819488, -0.804002648195039, 1.308642806941265, 2.544457077570663, 0.076191970411526, 4.199812227819406)$	35	18	2.017	3.536	-Do-
$(1.625618560729691, 1.780227435151377, 1.081125768865785, 1.929385970968730, 1.775712678608402, 1.486791632403172, 1.435858588580919, 1.446783749429806, 1.306349472016557, 1.508508655381127)$	8	5	1.958	3.199	$(1.843070932853103, 1.968335615552308, 1.619129623114350, 2.085033499005120, 2.563681448636798, 2.419409078868559, 2.715153752047097, 2.138630237161518, 2.568218081528525, 2.190731749055787)$

COC-Computational Order of Convergence   NM-Newton's Method
WFM-Weerakoon-Fernando Method   Tolerance = 1E-15